From Simple Math — Chaos

One of the nice things about math is that if you follow the rules you always get the same results. Well… most of the time.

Let's try an experiment. Let's calculate $3.7\,x(1-x)$ over and over again starting with $x = 0.5$. I've done it here using Excel on my laptop and a pocket calculator. (I was lazy and didn't copy all the calculator's numbers until they started getting interesting.)

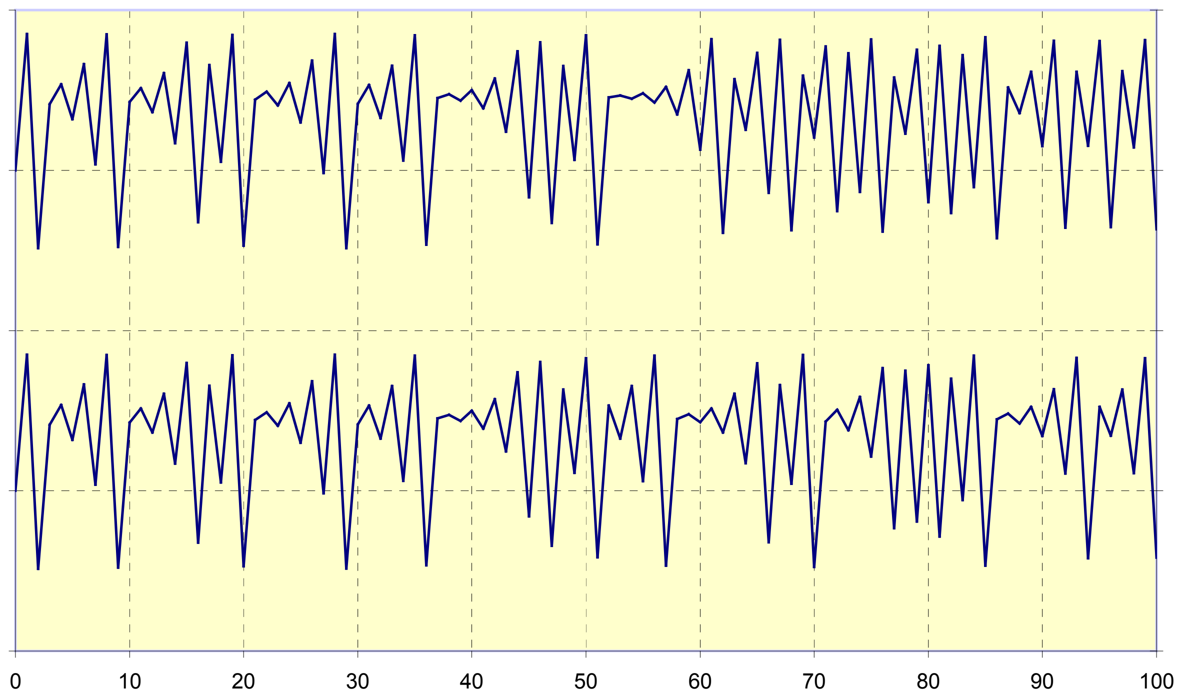
	Excel	Calculator
	0.50000	0.50000
1	0.92500	
2	0.25669	
3	0.70596	
4	0.76805	
5	0.65915	
6	0.83129	
7	0.51892	
8	0.92368	
9	0.26084	
10	0.71338	0.71338
11	0.75654	
12	0.68150	
13	0.80312	
14	0.58504	
15	0.89824	
16	0.33819	
17	0.82812	
18	0.52665	
19	0.92237	
20	0.26492	0.26492
21	0.72054	
22	0.74505	
23	0.70282	
24	0.77279	
25	0.64966	
26	0.84213	
27	0.49190	
28	0.92476	
29	0.25745	
30	0.70733	0.70733
31	0.76596	
32	0.66329	
33	0.82635	
34	0.53094	
35	0.92146	
36	0.26778	
37	0.72547	
38	0.73690	
39	0.71735	
40	0.75021	0.75023
41	0.69336	
42	0.78667	
43	0.62095	
44	0.87088	
45	0.41607	
46	0.89893	
47	0.33615	
48	0.82567	
49	0.53258	
50	0.92107	0.92144
51	0.26898	0.26783
52	0.72753	0.72555
53	0.73345	0.73677
54	0.72336	0.71758
55	0.74041	0.74983
56	0.71114	0.69406
57	0.76005	0.78566
58	0.67479	0.62307
59	0.81196	0.86896
60	0.56492	0.42131
61	0.90941	0.90209
62	0.30483	0.32681
63	0.78406	0.81401
64	0.62644	0.56016
65	0.86585	0.91161

Starting around the 40^{th} calculation, the number is different between Excel and the calculator. By the 50^{th} calculation the last two digits are different. On the 60^{th}, it's completely different.

So what's broken? My laptop or my calculator?

Nothing's broken. Computers and calculators cannot represent numbers exactly. My calculator can only represent 10 digits so there's a tiny difference between the exact value of a number and the calculator's representation of the number. Excel does a little better, it keeps about 16 digits.

This function we're exploring is a **Chaotic Function**. This means that its behavior is extremely dependent on its initial value. This chart shows how a very small change in the starting value leads to completely different behavior after about 50 steps.



Mathematical Chaos:
The top plot starts with 0.50000; the lower plot starts with 0.50001

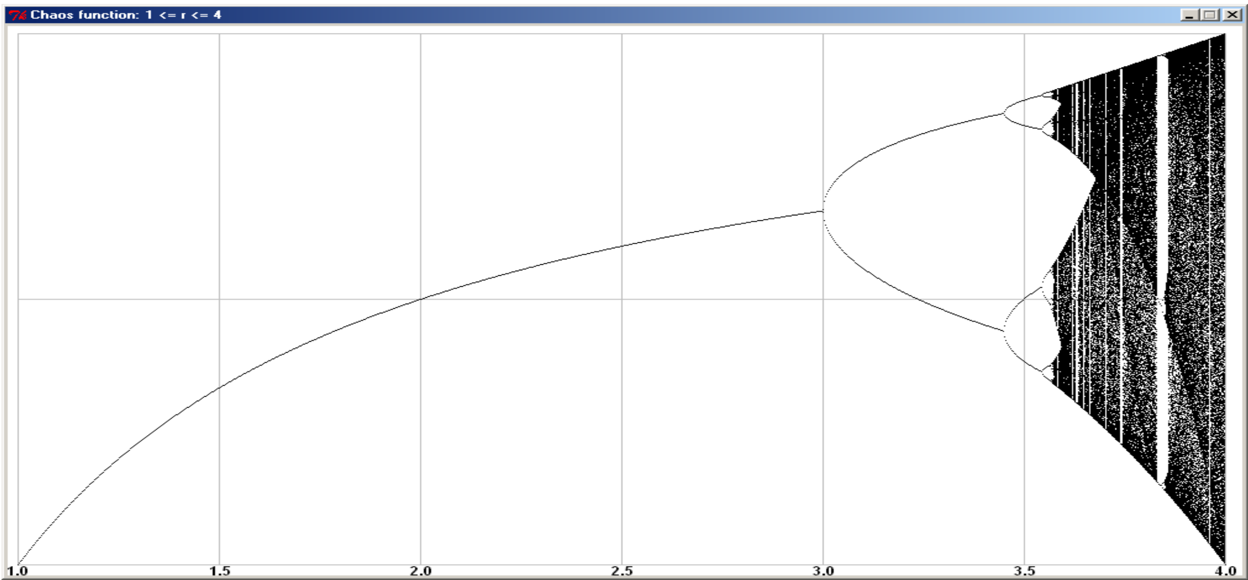
There is another interesting feature of this function. Replacing the constant 3.7 with the variable r gives us $r\,x(1-x)$. How does the behavior of the function change as r varies $1 \le r \le 4$. Again, using Excel, we can experiment.

	2	2.5	2.9	3	3.1	3.5	3.7	3.83	3.9
	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000
1	0.50000	0.62500	0.72500	0.75000	0.77500	0.87500	0.92500	0.95750	0.97500
2	0.50000	0.58594	0.57819	0.56250	0.54056	0.38281	0.25669	0.15586	0.09506
3	0.50000	0.60654	0.70727	0.73828	0.76990	0.82693	0.70596	0.50390	0.33550
4	0.50000	0.59662	0.60041	0.57967	0.54918	0.50090	0.76805	0.95744	0.86946
5	0.50000	0.60166	0.69576	0.73096	0.76750	0.87500	0.65915	0.15606	0.44263
6	0.50000	0.59916	0.61387	0.58997	0.55317	0.38282	0.83129	0.50443	0.96217
7	0.50000	0.60042	0.68740	0.72571	0.76624	0.82694	0.51892	0.95742	0.14197
8	0.50000	0.59979	0.62316	0.59716	0.55527	0.50088	0.92368	0.15612	0.47508
9	0.50000	0.60010	0.68102	0.72168	0.76553	0.87500	0.26084	0.50459	0.97258
10	0.50000	0.59995	0.62998	0.60257	0.55643	0.38282	0.71338	0.95742	0.10401
11	0.50000	0.60003	0.67601	0.71844	0.76513	0.82694	0.75654	0.15614	0.36345
12	0.50000	0.59999	0.63516	0.60686	0.55709	0.50088	0.68150	0.50464	0.90228
13	0.50000	0.60001	0.67202	0.71574	0.76490	0.87500	0.80312	0.95742	0.34387
14	0.50000	0.60000	0.63919	0.61036	0.55747	0.38282	0.58504	0.15615	0.87993
15	0.50000	0.60000	0.66882	0.71346	0.76476	0.82694	0.89824	0.50466	0.41204
16	0.50000	0.60000	0.64235	0.61330	0.55770	0.50088	0.33819	0.95742	0.94483
17	0.50000	0.60000	0.66624	0.71149	0.76468	0.87500	0.82812	0.15615	0.20331
18	0.50000	0.60000	0.64486	0.61582	0.55783	0.38282	0.52665	0.50466	0.63170
19	0.50000	0.60000	0.66414	0.70976	0.76463	0.82694	0.92237	0.95742	0.90736
20	0.50000	0.60000	0.64686	0.61801	0.55790	0.50088	0.26492	0.15615	0.32783
1000	0.50000	0.60000	0.65517	0.65917	0.55801	0.50088	0.75092	0.95742	0.83563
1001	0.50000	0.60000	0.65517	0.67399	0.76457	0.87500	0.69204	0.15615	0.53568
1002	0.50000	0.60000	0.65517	0.65918	0.55801	0.38282	0.78855	0.50467	0.97004
1003	0.50000	0.60000	0.65517	0.67398	0.76457	0.82694	0.61694	0.95742	0.11336
1004	0.50000	0.60000	0.65517	0.65919	0.55801	0.50088	0.87440	0.15615	0.39198
1005	0.50000	0.60000	0.65517	0.67398	0.76457	0.87500	0.40635	0.50467	0.92950
1006	0.50000	0.60000	0.65517	0.65920	0.55801	0.38282	0.89255	0.95742	0.25558
1007	0.50000	0.60000	0.65517	0.67397	0.76457	0.82694	0.35486	0.15615	0.74201
1008	0.50000	0.60000	0.65517	0.65920	0.55801	0.50088	0.84705	0.50467	0.74658
1009	0.50000	0.60000	0.65517	0.67396	0.76457	0.87500	0.47935	0.95742	0.73788
1010	0.50000	0.60000	0.65517	0.65921	0.55801	0.38282	0.92342	0.15615	0.75431

For $r < 3$ the repeated function settles to a constant value.
For $r = 3$ it looks like it's settling to a fixed value but very slowly.
For $r = 3.1$ it gets stuck in a loop with 2 values.
For $r = 3.5$ it gets stuck in a loop with 4 values.
For $r = 3.7$ it behaves chaotically.
For $r = 3.83$ it gets stuck in a loop with 3 values.
For $r = 3.9$ it behaves chaotically.

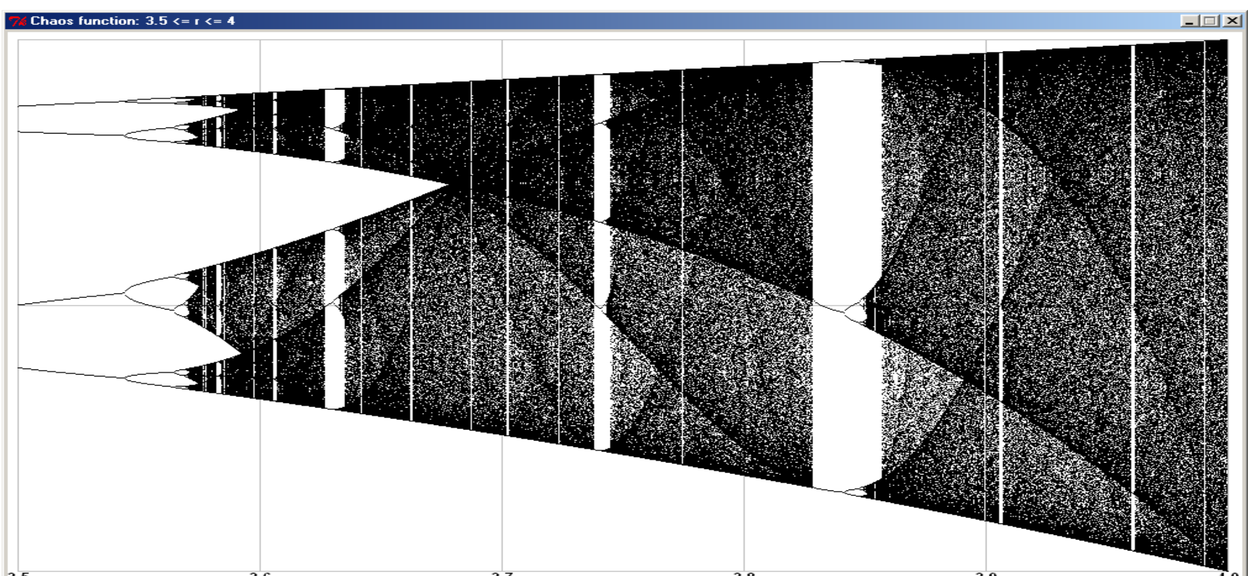
This is a really strange function. Changing *r* can dramatically change its behavior.

The behavior the function settles into is called an **attractor**. If we make a graph with *r* on the horizontal axis and a lot of the sequential values of the function on the vertical axis, we can quickly see where changes in behavior occur. The following graphs skip the first 10,000 values and then plot the next 1,000. This way we see the attractor uncluttered by the function values as they approach the attractor. Mathematicians call this kind of graph a **bifurcation diagram**.



Attractor for $r\,x(1-x)$

$1 \le r \le 4$



Attractor for $r\,x(1-x)$

$3.5 \le r \le 4$

It is easy to see on the detail graph that there are regions where the attractor is stable (either a fixed value or a multiple value cycle) and regions where it is chaotic.

Look at what happens when $r = 3$. The graph suddenly splits in two. Then at about $r = 3.55$ it splits in to four. It continues to double more and more often. If we looked close enough we'd see that when $r \approx 3.5699457$ the graph has split an infinite number of times and the function has become chaotic.

The black bands show where there is chaos and the white bands show where there is stability. It's interesting to see that there are regions of stability mixed in with the chaos.

If you look closely at the region around $r = 3.83$ you can see that each of the three branches follows the same doubling pattern that started at $r = 3.0$. So as r increases from 3.83 the attractor moves from 3-cyclic to 6-cyclic to 12, 24, 48…-cyclic and then breaks back to chaos.

