MIXING APPLES AND ORANGES

In the 17th century Brook Taylor determined how to generate a "power series" expansion for almost any function. Here are the Taylor series expansions for some functions we have been working with.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

1) Write the first eight terms of the series expansion for e^{ix} by substituting (ix) for x in the first equation.

2) Expand the $(ix)^n$ terms and evaluate the powers of i. Note that there is a pattern to the values of i^n.

3) Group the real terms together and then group the imaginary terms together. The result should look like $e^{ix} = (terms...) + i(terms...)$

If you've done everything correctly, the real terms should be the series expansion for cos x and the imaginary terms should be the series expansion for sin x. Thus we can write

$$e^{ix} = \cos x + i \sin x$$

This is known as *Euler's formula* (pronounced **oil**-er), discovered in 1748 by Leonhard Euler. This amazing formula lets us write one of the more beautiful equations in all of math

$$e^{i\pi} + 1 = 0$$

which ties together five important constants.

4) What is the formula for e^{a+bi}? Hint: use one of the power rules first and then use Euler's formula.

AND YOU THOUGHT TRIGONOMETRY WAS COMPLEX

Using Euler's formula you can derive formulas for sin *ix* and cos *ix* in terms of e^x.

5) What is the formula for e^{-ix}? Hint: remember that sine is an odd function and that cosine is an even function.

6) Use the formulas for e^{ix} and e^{-ix} to solve for sin *x* and cos *x*. Hint: treat this as two equations in two unknowns where the unknowns are (sin *x*) and (cos *x*).

7) Replace *x* with (*ix*) in the sin *x* and cos *x* formulas to get formulas for sin *ix* and cos *ix*.

To derive formulas for $\sin(a+bi)$ and $\cos(a+bi)$ you are going to need the sine and cosine angle addition formulas. In case you've forgotten them, they are easily derived using Euler's formula by substituting (*a*+*b*) for *x*:

$$\cos(a+b)+i\sin(a+b)=e^{i(a+b)}$$
$$\cos(a+b)+i\sin(a+b)=e^{ia}e^{ib}$$
$$\cos(a+b)+i\sin(a+b)=(\cos a+i\sin a)(\cos b+i\sin b)$$
$$\cos(a+b)+i\sin(a+b)=\cos a\cos b+i\cos a\sin b+i\sin a\cos b+i^2\sin a\sin b$$
$$\cos(a+b)+i\sin(a+b)=\cos a\cos b-\sin a\sin b+i(\cos a\sin a+\sin a\cos b)$$

Since the real and imaginary parts of a complex number do not interact with each other, we can separate the last equation into two. We can also divide out the *i* in the expression for $\sin(a+bi)$.

$$\sin(a+b)=\sin a\cos b+\cos a\sin b$$
$$\cos(a+b)=\cos a\cos b-\sin a\sin b$$

Mathematicians are lazy but have good memories. When they use the same equations over and over they give them names. Here are formulas for the *hyperbolic* sine, cosine and tangent:

$$\sinh x=\frac{e^x-e^{-x}}{2},\qquad \cosh x=\frac{e^x+e^{-x}}{2},\qquad \tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

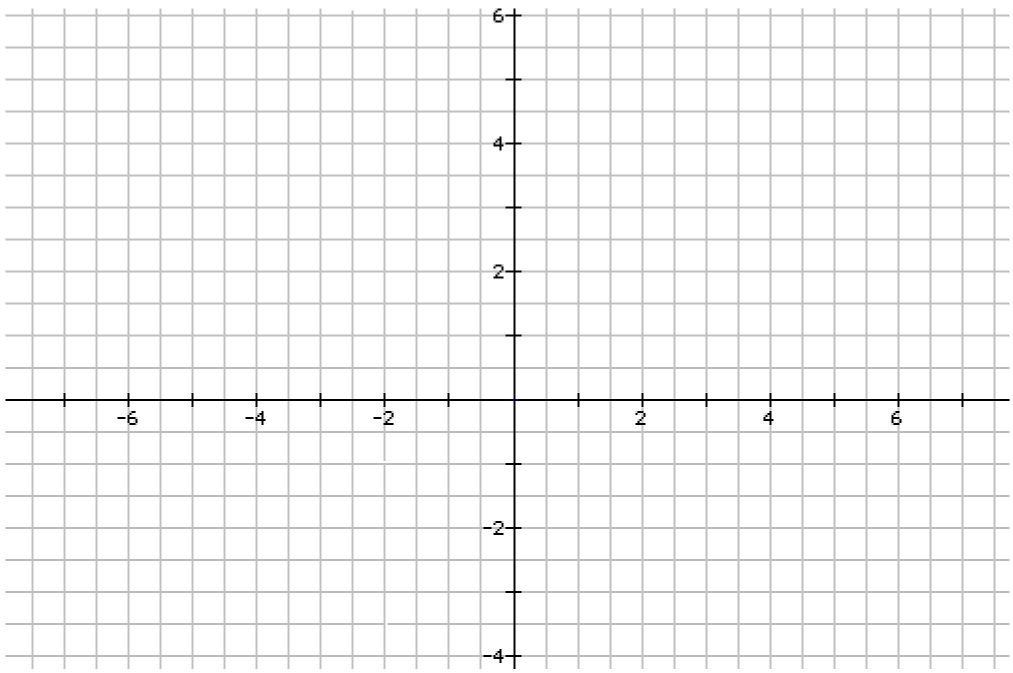
These let you write sin *ix*, cos *ix* and tan *ix* as

$$\sin ix=i\sinh x,\qquad \cos ix=\cosh x,\qquad \tan ix=i\tanh ix$$

8) What is cot *ix* in terms of coth *x*?

9) Derive formulas for $\sin(a+bi)$ and $\cos(a+bi)$

10) Graph the sinh, cosh, and tanh functions.



RANGES? WE DON'T NEED NO STINKIN' RANGES!

As you can see, the real part of $\sin(a+bi)$ can be infinitely large. This suggests that $\sin^{-1} 2$ should exist as a complex number. You need to look for solutions to

$$2 = \sin(a+bi)$$
$$2 = \sin a \cosh b + i \cos a \sinh b$$

11) The imaginary term must be 0. For what values of a and b will $\cos a \sinh b = 0$?

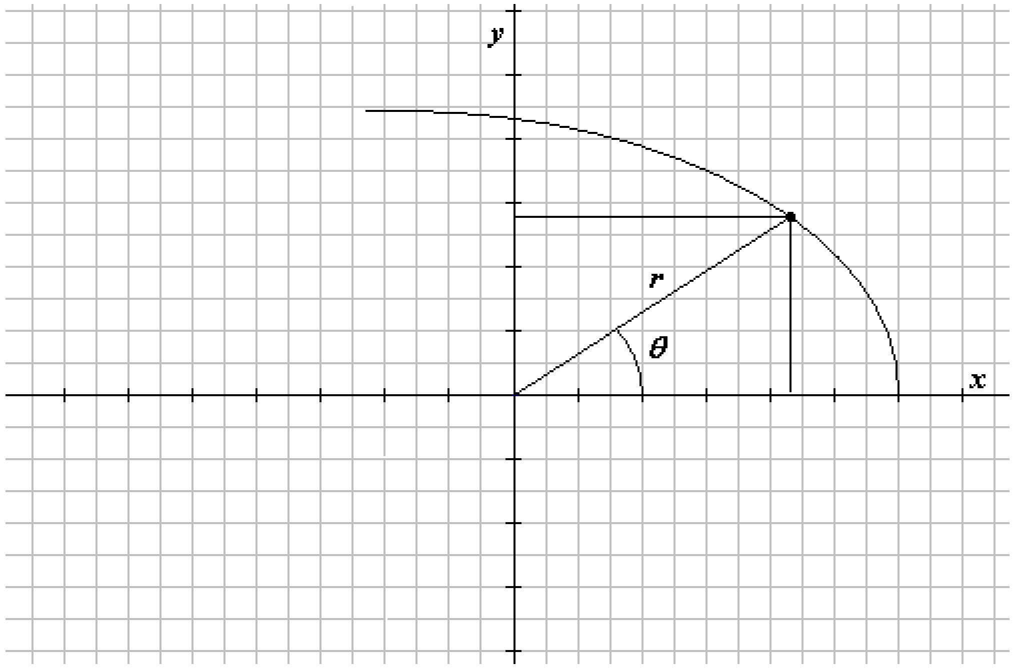
12) Given the constraints on the values of a and b, what is to solution to $\sin a \cosh b = 2$?

13) The a and b you just found are the real and imaginary components to the arcsine of 2!

$\sin^{-1} 2 =$

ARCTIC EXPLORATION

Do you remember polar equations and polar graphs?



The conversion from polar form (r,θ) to rectangular form (x,y) is

$$x = r\cos\theta$$
$$y = r\sin\theta$$

If we replace the x axis with the real axis and the y axis with the imaginary axis, we can locate points on the complex plane with polar coordinates. (Here's another lazy mathematician shortcut: $z = a + bi$ then $\text{re}(z) = a$ and $\text{im}(z) = b$.)

$$\text{re}(z) = r\cos\theta$$
$$\text{im}(z) = r\sin\theta$$
$$z = r\cos\theta + i\,r\sin\theta$$
$$z = r(\cos\theta + i\sin\theta)$$
$$z = r\,e^{i\theta}$$

This gives us a compact way to write complex numbers in polar form that is easy to manipulate algebraically.

14) What is $r_1e^{i\theta_1}$ times $r_2e^{i\theta_2}$? Express your answer as a complex number in polar form.

15) What is $\left(r\,e^{i\theta}\right)^n$? Express your answer as a complex number in polar form.

16) Why might we want to work with complex numbers in polar form? Think about computing $\left(1+i\sqrt{3}\right)^{100}$ versus $\left(2\,e^{i\pi/3}\right)^{100}$. What is the value of $\left(2\,e^{i\pi/3}\right)^{100}$?

17) Can you think of geometrical rules for multiplication and exponentiation of complex numbers in polar form?

18) What is $\sqrt[n]{r\,e^{i\theta}}$? Remember that there are *n* solutions. Hint: in polar coordinates $\theta \equiv \theta + 2\pi \equiv \theta + 4\pi \equiv \cdots$

THE POWER OF IMAGINATION

What is i^i ?

19) Think about the exponentiation rule for complex numbers. What is *i* in polar form? What is that value raised to the power of *i*?

20) Remember that the polar form of a complex number is periodic in 2π . i^i is multi-valued. In fact, there are an infinite number of solutions. What are they?

