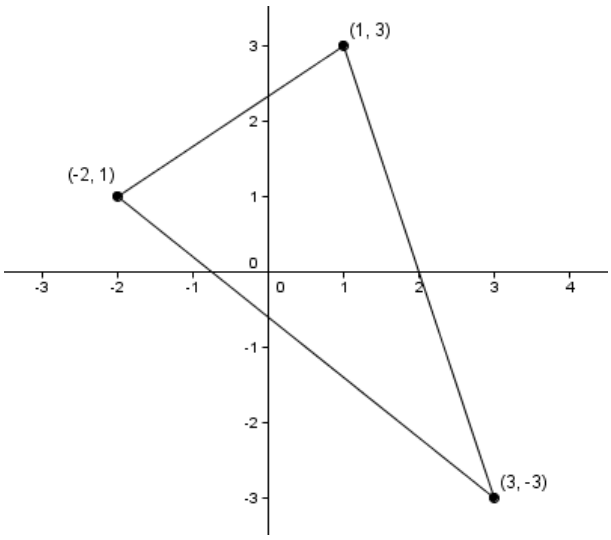
Introduction

Normal graph paper is covered with same-sized squares, and when you use it in math you usually choose one of the horizontal lines to be the *x*-axis and one of the vertical lines to be the *y*-axis. After drawing scales on the *x*- and *y*-axes you can locate any point on the graph paper by its *x* and *y* coordinates. The coordinates are usually written as (*x-coordinate*, *y-coordinate*). Here's an example showing a triangle with coordinates (-2, 1), (1, 3) and (3, -3).



Drawing a map

You probably live in a city that primarily has rectangular blocks. The streets go east/west and north/south. This makes it pretty easy for you to draw a map of your town on a regular piece of graph paper. The main east/west street goes on the *x*-axis with east at the right end of the axis, and the main north/south street goes on the *y*-axis with north at the top.

Starting at the center of town, it's easy to say, "go two blocks west and one block north and you'll find the west-most corner of Triangle Park."

Now, let's move the town to the North Pole. Tell the polar bear that's sitting at the center of town how to get to the corners of Triangle Park. "Go west, uh…" Do you see the problem? From the North Pole, there is no direction but south!

The Polar Bear's Map

This polar bear is really smart. He knows where the Prime Meridian (zero degree line of longitude) is and can measure angles to find any other meridian. You can tell him where things are by telling him how far to travel from the North Pole along which meridian. Every point that you can name using direction coordinates in your home town can be located in the polar bear's world by distance and angle coordinates.

Here's Triangle Park located on the polar bear's map. All the streets really go all the way to the North Pole, but they get too close together to show on the map.

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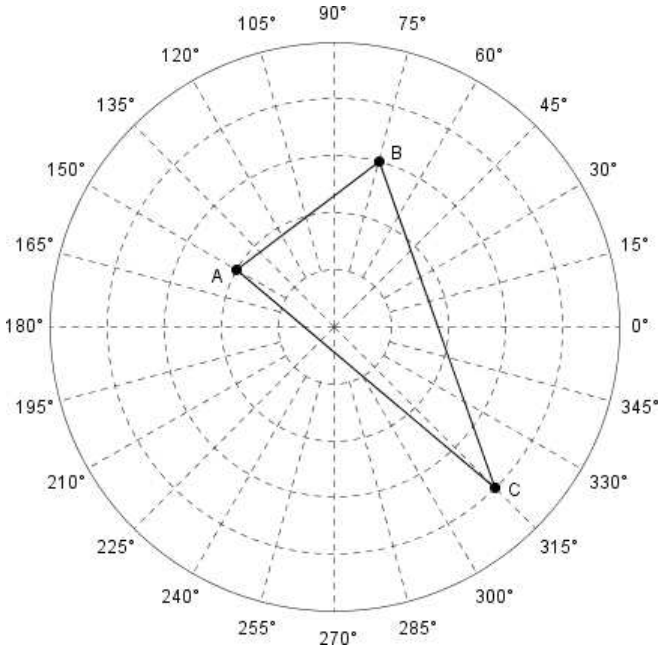

The Polar Bear's Map

A = 2 blocks at 150°, B = 3 blocks at 75°, C = 4 blocks at 315°

Mathematically these are called *polar coordinates* and they are written as $r\angle\theta$ where r is the distance and θ (Greek letter theta) is the angle. The corners of Triangle Park are $2\angle 150°$, $3\angle 75°$, and $4\angle 315°$.

(The regular kind of coordinates are called *rectangular*, or less commonly, *Cartesian* coordinates.)

How Many Address Do You Have?

In rectangular coordinates, houses only have one address. For example, if you live two blocks west and 3 blocks north of the town center your address is (-2, 3). Any other combination of numbers is the address for somewhere else.

What happens if we ask the polar bear to face the 30° meridian and then walk backwards two blocks? We've asked him to go to $-2\angle 30°$. This is exactly the same place as $2\angle 210°$! Every point on the polar bear's map has two addresses.

But wait, there's more; not sold in any store. The polar bear has always measured counter-clockwise to find the direction he needs to travel. What if we give him a negative angle? Just like negative numbers go the other way on the number line, negative angles turn the other direction. Let's send him to $2\angle -150°$. Turning clockwise 150° is the same direction as turning counter-clockwise 210°; once again he's arrived at $2\angle 210°$. Now we're up to four address for every point on the map.

Order before midnight and we'll include this amazing free gift. If we ask the polar bear to measure more than 360° he keeps turning in circles until he gets to the required direction. To get

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to 570° he has to turn a full circle plus 210° so $2\angle 570°$ is another address for $2\angle 210°$. Add or subtract as many multiples of 360° as you want to the angle in an address and the polar bear gets to the same spot. Every point on the polar map has an infinite number of addresses.

The polar bear doesn't like getting dizzy so he figured out that he can take any big angle and divide by 360° and use the remainder to determine how far he needs to turn. For instance $1234° / 360° = 3\,\mathrm{r}154°$ so he can turn 154° and stop instead of spinning around nearly 3 1/2 times.

Converting Between Coordinate Systems

When we moved triangle park to the north pole, we cheated a bit and moved the corners so they'd land on the polar bear's street corners. The conversion between rectangular and polar coordinates does not come out exactly. The actual numbers came out:

Rectangular	Polar
(-2, 1)	$2.23607\angle 153.43495°$
(1, 3)	$3.16228\angle 71.56505°$
(3, -3)	$4.24264\angle 315°$

Converting between rectangular and polar coordinates requires trigonometric functions. Usually, when polar graphing is used the data that is displayed is naturally based on angles and does not need to be converted back and forth to rectangular coordinates.

If you do need to convert between coordinate systems, here are the formulas to use.

Polar to Rectangular	Rectangular to Polar
$x = r\cos\theta$ $y = r\sin\theta$	$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\dfrac{y}{x}$

Computing the θ value is a little tricky because there are some special cases:

$$\text{When } x = 0 \text{ then } \theta = \begin{cases} 90°, & y > 0 \\ 0°, & y = 0 \\ 270°, & y < 0 \end{cases}$$

$$\text{When } x \neq 0 \text{ then } \theta = \begin{cases} \tan^{-1}(y/x), & x > 0,\ y \geq 0 \\ \tan^{-1}(y/x) + 360°, & x > 0,\ y < 0 \\ \tan^{-1}(y/x) + 180°, & x < 0 \end{cases}$$

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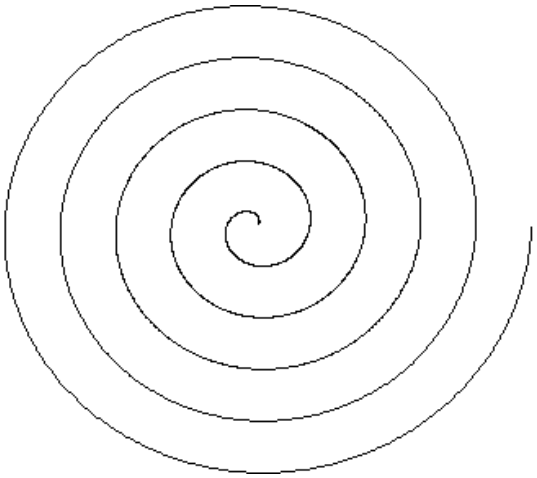
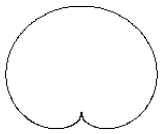
Plotting Functions on Polar Graphs

Just like you can plot functions on regular graph paper, you can plot functions on polar graph paper. The function $y = Ax$ is a straight line on rectangular graph paper; it's a spiral on polar graph paper. There is no reason we need to limit θ to the range [0, 360°]; angles go on forever, they just keep going around the polar plot. They can also be negative. Unless otherwise specified, $0 \le \theta \le 360°$.

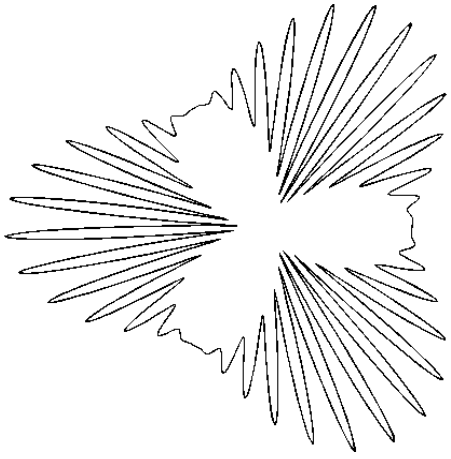
$r = \theta / 360$
$0 \le \theta \le 1800°$



$r = 1 + \sin\theta$



$r = 2.5 + (\sin 40\theta)(1 - \cos 3\theta)$



You can get some amazingly pretty results when you experiment with sin and cos functions!

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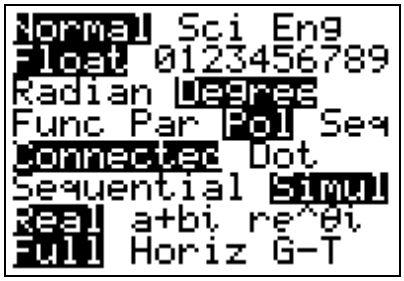
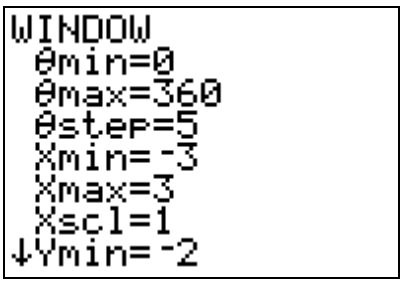
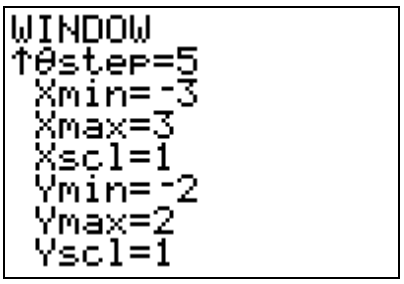
Polar Plotting with TI-83/84 Calculators

Set your calculator in "degree" mode and "polar" graphing mode (MODE key)

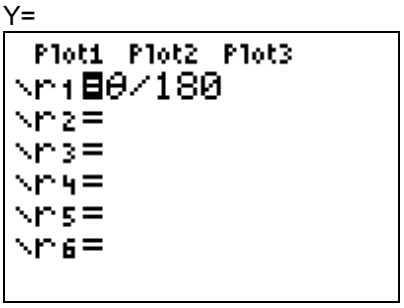


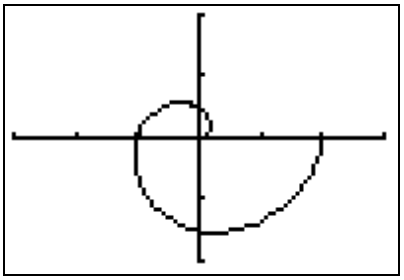
Set the graph window parameters (WINDOW key)





Graph a few functions, for example





Try some other functions, for example $r = \sin(N\theta) + C$ when N is an integer and C is any number.

Add a cosine into the mix, try $r = \sin(2\theta) + \cos(3\theta) + 1.5$. (This is a bit too big to show with the window settings from above; use ZOOM FIT, ZOOM SQUARE to see it.)

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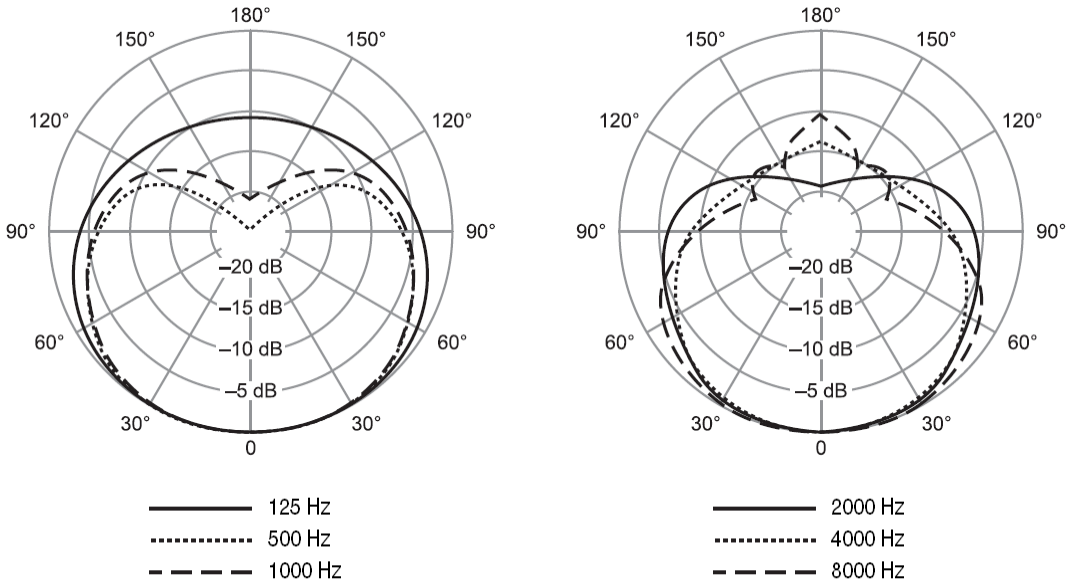
Where Are Polar Coordinates and Graphs Used in the Real World?

Radar measures the things it is tracking by the angle and distance from the antenna. This maritime navigation radar shows the ship's position in the center of the display and other ships and land as distance and direction from the ship. (The straight lines attached to the other ships are computerized course projections for those ships.)



TYPICAL POLAR PATTERNS

The pickup pattern of microphones is shown on polar plots. This plot shows the pattern for one of the most popular directional microphones used for vocal performance. The pattern is called "cardioid," (meaning heart-shaped). You can see that the shape changes a bit depending on the sound's frequency. Each circle on the plot represents a bit less than doubling of the sound reception, so this microphone "hears" things about 10 times better from the front than from the back.

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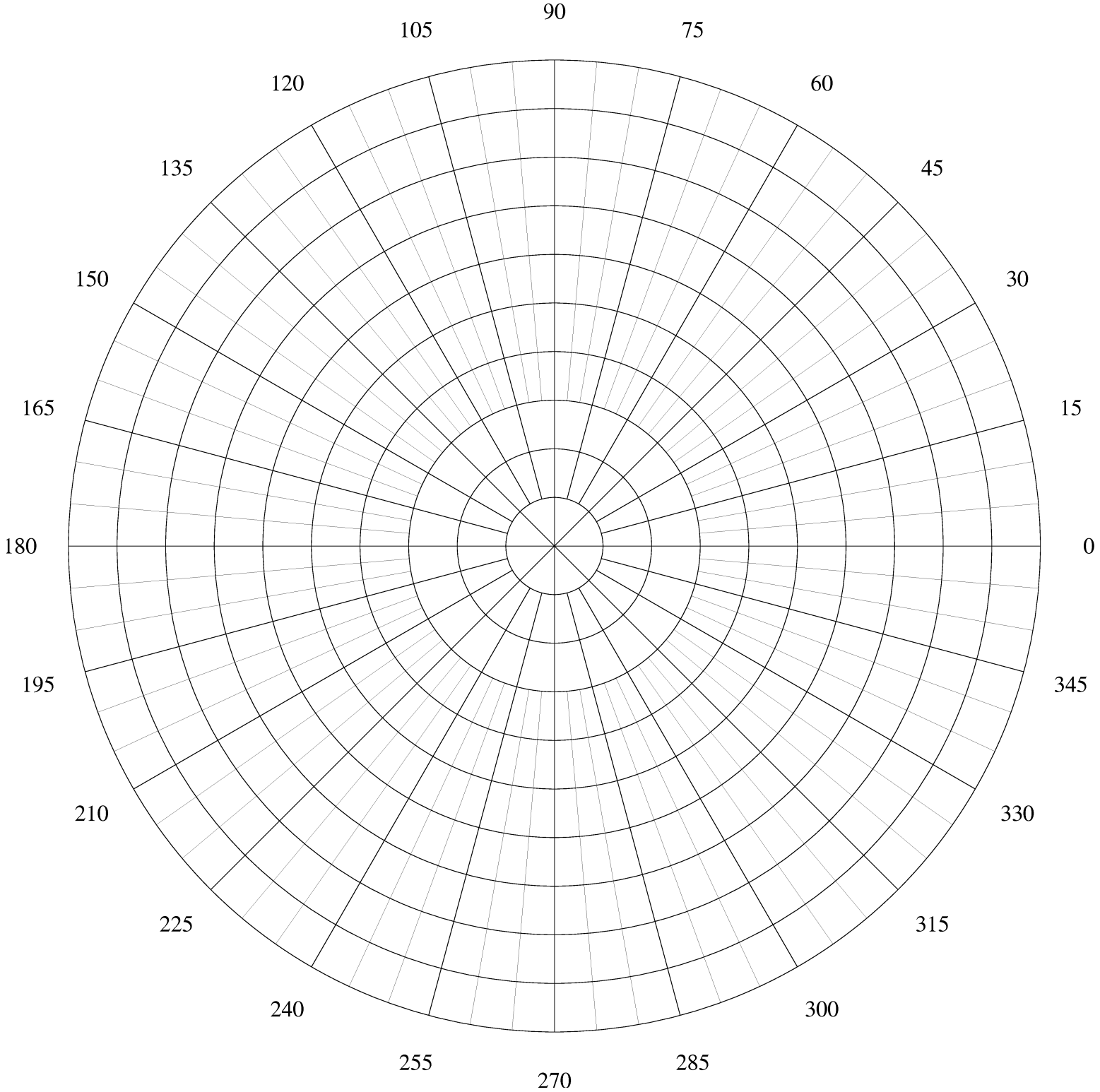
90
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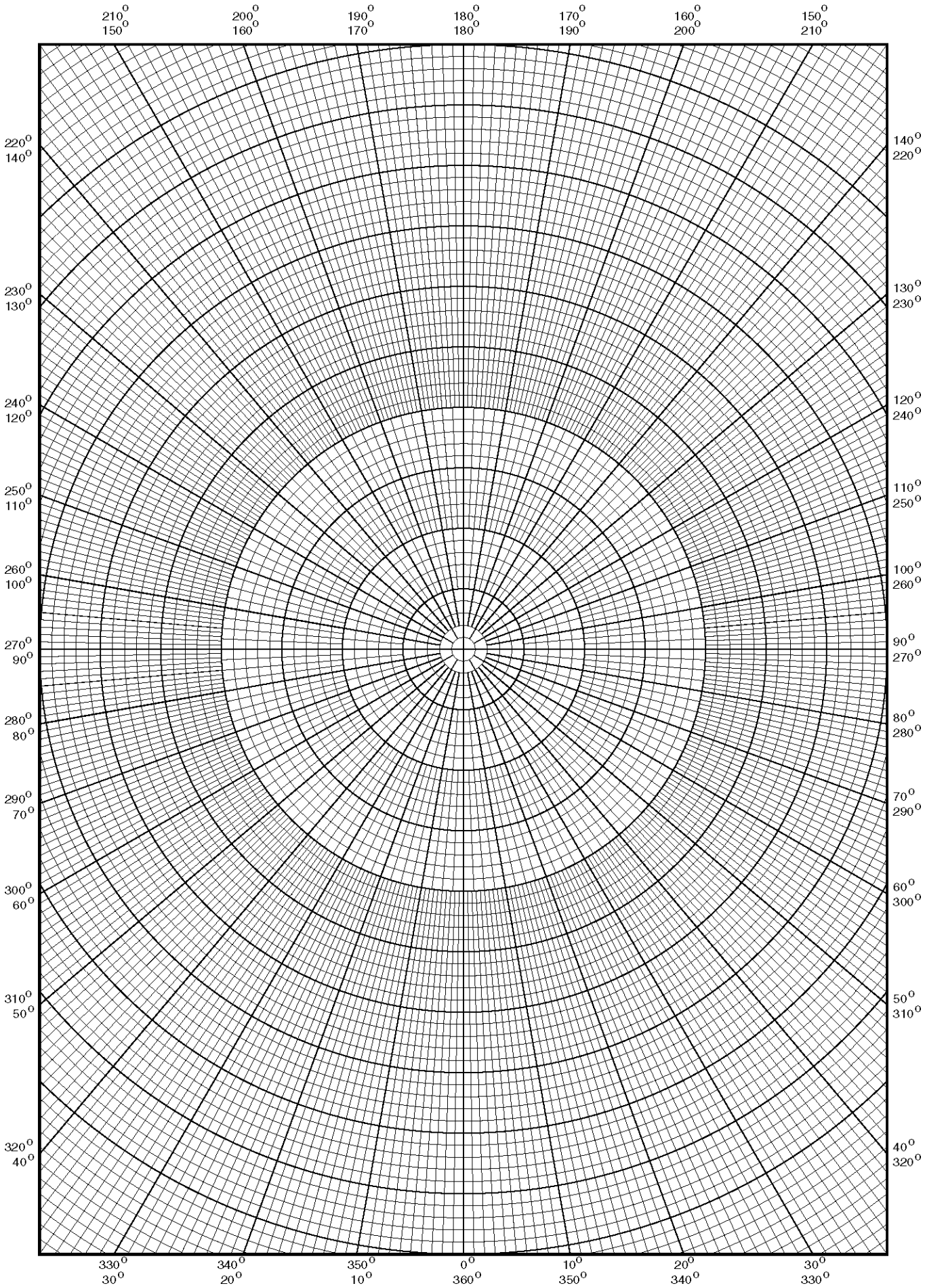
210° 150°
200° 160°
190° 170°
180° 180°
170° 190°
160° 200°
150° 210°
220° 140°
230° 130°
240° 120°
250° 110°
260° 100°
270° 90°
280° 80°
290° 70°
300° 60°
310° 50°
320° 40°
140° 220°
130° 230°
120° 240°
110° 250°
100° 260°
90° 270°
80° 280°
70° 290°
60° 300°
50° 310°
40° 320°
330° 30°
340° 20°
350° 10°
0° 360°
10° 350°
20° 340°
30° 330°